Stochastic Geometric Modeling of Wireless Communication Networks

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BRITISH COUNCIL

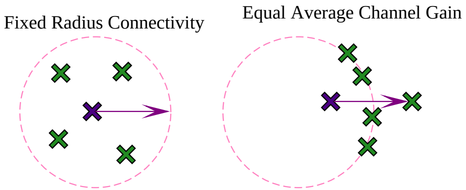
RESEARCHER LINKS

Outline

Outline

The Paradigm Shift

- Wireless Networks:
  - Protocol design & optimization
  - Performance Analysis
- Traditional approach:
  - Point-to-point analysis
  - Fixed topology : Deterministic Graphs



Limitations

We have all learned to draw a graph to depict a communication network, as in Fig.1. This is a useful and accurate depiction of the network topology when the nodes are interconnected with dedicated wired lines. The tendency has been to do the same when the network under consideration is a wireless one, and that has been the cause of many misconceptions and much fallacious reasoning. If there are no "hard-wired" connections between the nodes, the notion of a "link" between, say, nodes A and B is an entirely relative one. In fact, it is so relative that links in a wireless network should be thought of as "soft" entities that are almost entirely under the control of the network operator.

⋮

It should be clear, then, that the existence of a wireless link is a very volatile notion. Thus, the proper way of depicting a wireless network is simply via the location of its node

-Ephermides

Spatial Stochastic Models

- Point pattern based location modeling.
- Infinite realizations of point process;
- Impractical to design by considering infinite network topologies;
- A statistical point pattern $\equiv$ Spatial stochastic models.

Definition

Stochastic geometry is the branch of mathematics which deals with the study of random point processes .

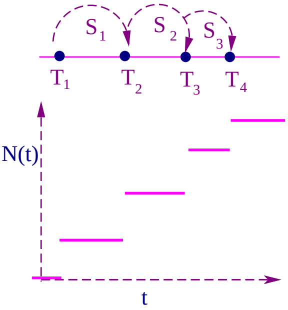
Outline

Applications of Stochastic Geometry

- General applications:
  - Astronomy
  - Sterelogy
  - Forestry
  - Material Sciences
  - Pattern extraction
- Wireless Communication:
  - Interference Modeling
  - Cooperative transmission & sensing
  - Security
  - Microwave power transfer
  - Sensor networks, IoT, D2D and M2M
  - Vehicular networks
  - Energy harvesting

Point process

- Arrival times: $T_{i+1} < T_i \rightarrow$Study inter-arrival $S_i = T_{i+1} - T_i$;[1]
- Counting process $N_t = \sum_{i=1}^{\infty} \mathbb{1}\{T_i \leq t\}$;



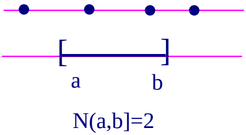


Figure : Arrival times, inter-arrivals and counting process.

[1]In this tutorial, we follow the text by Baddeley 2007 (full reference at the end).

Point Process

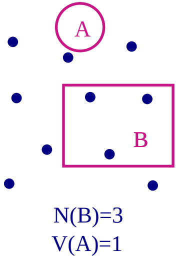


Figure : Count and vacancy indicators

- $N(B) = \#$ number of points in $B \in \mathbb{R}^2$.
- $V(A) = \mathbb{1}\{N(A) = 0\}$.
- $N(B)$ is natural for exploring additive properties
  - e.g. $N(B) = N_{red}(B) + N_{black}(B)$.
- $V(A)$ are natural for exploring geometric and 'multiplicative' properties
  - e.g. $V(A) = V_{red}(A)V_{black}(A)$.

Binomial Point Process

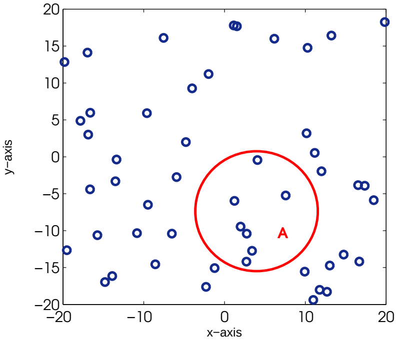


Figure : Binomial Point Process with $n = 50$ and $\mathscr{W} = L \times W = 40 \times 40$.

- Fixed number of point n at random location inside $\mathscr{W} \subset \mathbb{R}^2$.
- Point process $\Phi = \{X_i\}\, i = 1, 2, \ldots, n$ with X_i i.i.d. uniformly distributed in $\mathscr{W}$.

$$f(\mathbf{x}) = \begin{cases} 1/v_2(\mathscr{W}) & \text{if } \mathbf{x} \in \mathscr{W} \\ 0 & \text{otherwise.} \end{cases} \qquad (1)$$

- $v_2(\mathscr{W}) = \int_{\mathscr{W}} d\mathbf{x} = L \times W$
- For a bounded set $\mathscr{B} \subseteq \mathscr{W} \subset \mathbb{R}^2$, the probability

$$p = \mathbb{P}(X_i \in B) = \frac{v_2(\mathscr{B} \cap \mathscr{W})}{v_2(\mathscr{W})}. \qquad (2)$$

Binomial Point Process

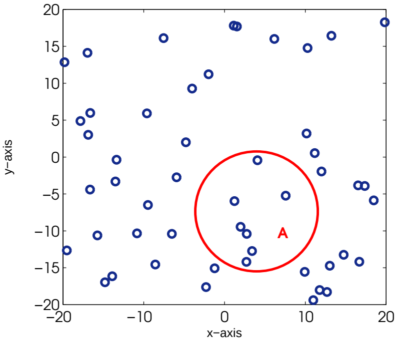


Figure : Binomial Point Process with $n = 50$ and $\mathscr{W} = L \times W = 40 \times 40$.

- $N(\mathscr{B}) = \sum_{i=1}^{n} \mathbb{1}\{X_i \in \mathscr{B}\}$

$$\mathbb{P}(\Phi(\mathscr{B}) = m) = \begin{pmatrix} n \\ m \end{pmatrix} p^m (1-p)^{n-m}. \quad (3)$$

- where $p = \mathbb{P}(X_i \in \mathscr{B}) = \frac{v_2(\mathscr{B} \cap \mathscr{W})}{v_2(\mathscr{W})}$ and $N(\mathscr{B})$ follows binomial distribution.
- $V(\mathscr{B}) = \mathbb{1}\{N(\mathscr{B}) = \emptyset\} = (1-p)^n$.
- Let $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ and $\mathscr{B}_1 \cap \mathscr{B}_2 = \emptyset$ then

$$\begin{aligned} N(\mathscr{B}) &= N(\mathscr{B}_1) + N(\mathscr{B}_2) \\ &= N(\mathscr{B}_1 \cup \mathscr{B}_2) \leq n. \end{aligned}$$

Distance Distribution for BPP

- Let $\mathscr{W} = b(o, R_c)$ i.e., ball of radius R_c centered at origin, then

$$\begin{aligned} \mathscr{F}_{R_i}(r) &= 1 - \mathbb{P}\{R_i \geq r\} \\ &= 1 - \sum_{k=0}^{i-1} \binom{n}{k} p^k (1-p)^{n-k} \\ &= 1 - I_{1-p}(n-i+1, i). \end{aligned} \tag{4}$$

where $p = r^2/R_c^2$ and $I_x(a,b) = \frac{\int_0^x t^{a-1}(1-t)^{b-1} dt}{\mathscr{B}(a,b)}$ is incomplete regularized beta function.

- $\mathscr{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function.

$$\begin{aligned} f_{R_n}(r) = \frac{2}{R_c} \frac{\Gamma(i+1/2)\Gamma(n+1)}{\Gamma(i)\Gamma(n+3/2)} \\ \times \beta\left(\left(\frac{r}{R_c}\right)^2; i+\frac{1}{2}, N-i+1\right). \end{aligned} \tag{5}$$

Distance Distribution for BPP

$$\beta\left(x;a,b\right)=\frac{x^{a-1}\left(1-x\right)^{b-1}}{\mathscr{B}(a,b)}. \tag{6}$$

- Application to wireless network
  - why is the n^{th} node forwarding important?
  - distance distribution to farthest and closest nodes
  - long hop vs short hop
  - energy efficiency tradeoffs
- What is missing?

Random Measure and Random Set Formalism

- $N(\mathscr{A})$ for all $\mathscr{A} \in \mathbb{R}^2$ provides sufficient information to reconstruct the positions in point pattern.
- $N(\{x\}) > 0$ forms point pattern; It is defined as $N(\mathscr{A})$ random variable indexed by $\mathscr{A}$.
- $N(\mathscr{A} \cup \mathscr{B}) = N(\mathscr{A}) + N(\mathscr{B})$ whenever $\mathscr{A} \cap \mathscr{B} = \emptyset$ and $N(\emptyset) = 0$.
- $\mathscr{A}_n \supseteq \mathscr{A}_{n+1}$is decreasing sequence of closed, bounded sets $\cap_n \mathscr{A}_n = \mathscr{A}$ then $N(\mathscr{A}_n) \to N(\mathscr{A})$.
- $N(\mathscr{A})$ is a random measure.
- Locally finite: $N(\mathscr{A}) < \infty$ with probability 1
- Simple:$N(\{x\}) \leq 1$ for all $x \in \mathbb{R}^d$.

Stationary Point Process

Stationary Point Process

A point process is stationary if its distribution is invariant under translation.

- For a stationary process, analyzing the performance of so called typical node is sufficient for networkwide characterization.
- Under stationarity the translated point process is identical to the original process.
- Binomial point process is not a stationary point process.
- A stationary point process cannot be defined in finite bounded compact subset of $\mathbb{R}^2$.

Isotropic

A point process is isotropic if its distribution does not change under the rotation operation.

- Motion invariant = isotropic + stationary

Poisson Point Process

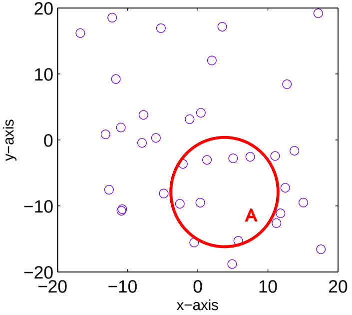


Figure : Poisson Point Process with $\lambda = 10^{-1}$ and $\mathscr{W} = L \times W = 40 \times 40$.

- Most popular model of spatial node locations
  - Rayleigh fading equivalent for point processes
  - Analytically tractable
- Statistical independence of node counts in disjoint subsets of $\mathbb{R}^2$.
- Random number of nodes
- Defined over whole plane
- Limitations: Any Guesses?

Poisson Point Process

Poisson point process

The spatial Poisson point process, with uniform intensity $\lambda > 0$, is a point process in $\mathbb{R}^2$ such that

- [PP1] for every bounded closed set $\mathscr{B}$, the count $N(\mathscr{B})$ has a Poisson distribution with mean $\lambda v_2(\mathscr{B})$.
- [PP2] if $\mathscr{B}_1, ..., \mathscr{B}_m$ are disjoint regions, then $N(\mathscr{B}_1), ..., N(\mathscr{B}_m)$ are independent.

Generalizing the above process yields

$$\mathbb{P}(\Phi(\mathscr{B}) = m) = \frac{\Lambda(\mathscr{B})^m}{m!} \exp(-\Lambda(\mathscr{B})). \tag{7}$$

with

$$\Lambda(\mathscr{B}) = \int_{\mathscr{B}} \lambda(\boldsymbol{x}) d\boldsymbol{x}. \tag{8}$$

Conditional Property

Conditional Property

Consider a spatial Poisson point process, with uniform intensity $\lambda > 0$ in $\mathbb{R}^2$. Let $\mathscr{W} \subset \mathbb{R}^2$ be any finite region with $0 < v_2(\mathscr{W}) < \infty$. Given that $N(\mathscr{W}) = n$, the conditional distribution of $N(\mathscr{B})$ for $\mathscr{B} \subset \mathscr{W}$ is binomial:

$$\mathbb{P}(N(\mathscr{B}) = k | N(\mathscr{W}) = n) = \binom{n}{k} p^k (1-p)^{n-k}, \tag{9}$$

where $p = v_2(\mathscr{B})/v_2(\mathscr{W})$. Furthermore the conditional joint distribution of $N(\mathscr{B}_1), ..., N(\mathscr{B}_m)$ for any $\mathscr{B}_1, ..., \mathscr{B}_m \subset \mathscr{W}$ is the same as the joint distribution of these variable in binomial point process.

Distance Distribution of Poisson Point Process

- Distance to the n^{th} neighbour in Poisson point process

$$\mathbb{P}(R_n < r) = \frac{\gamma(n, \lambda \pi r^2)}{\Gamma(n)}. \tag{10}$$

- Gamma distributed random variable.
- The average distance to the n^{th} neighbour can be found as

$$\mathbb{E}(R_n) = \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi}\lambda\Gamma(n)}.$$

Campbell's Theorem

Sum of function over a PPP

Let Φ be a Poisson point process on $\mathbb{R}^2$ with intensity λ and let $f : \mathbb{R}^2 \to \mathbb{R}^+$ be a measurable function. Then the random sum

$$S = \sum_{x \in \Phi} f(x), \tag{11}$$

is a random variable with

$$\mathbb{E}\left(\sum_{x \in \Phi} f(x)\right) = \lambda \int f(x) dx. \tag{12}$$

Campbell's Theorem

To prove the Campbell's theorem, consider

$$f(x) = \sum_i c_i \mathbb{1}\{x \in B_i\} \tag{13}$$

then the expected value of summation can be computed as

$$\begin{aligned}
\mathbb{E}(S) &= \mathbb{E}\left[\sum_{x\in\Phi} f(x)\right], \qquad (14)\\
&= \mathbb{E}\left[\sum_{x\in\Phi}\sum_i c_i \mathbb{1}\{x \in B_i\}\right] = \mathbb{E}\left[\sum_i c_i N(B_i)\right]\\
&= \sum_i c_i \Lambda(B_i) = \sum_i \int_{B_i} c_i \lambda(dx)\\
&= \int_B f(x)\lambda dx.
\end{aligned}$$

Probability Generating Functional

Product of function over a PPP

Let Φ be a Poisson point process on $\mathbb{R}^2$ with intensity λ and let $f : \mathbb{R}^2 \to [0,1]$ be a real valued function. Then

$$\mathbb{E}\left(\prod_{x\in\Phi} f(x)\right) = \exp\left(-\lambda \int_{\mathbb{R}^2} (1 - f(x))\, dx\right). \tag{15}$$

Outline

Mean & Variance of Aggregate Interference

- Consider a receiver located at the origin $\boldsymbol{o}$ associated with a transmitter at a fixed distance say r_o. The aggregate interference experienced by the receiver is given as

$$I = \sum_{i \in \Phi} h_i l(r_i) \tag{16}$$

- where h is unit mean exponential random variable corresponding to Rayleigh fading and $l(r)$ is the path-loss attenuation function.
- The average aggregate interference can be computed by employing the Campbell's theorem as

$$\begin{aligned} \mathbb{E}(I) &= \mathbb{E}(h)\lambda \int_0^{2\pi} \int l(r) r dr d\theta. \\ &= \lambda 2\pi \int_0^{\infty} r l(r) dr. \end{aligned} \tag{17}$$

- The variance can be computed along the same lines.

Link Outage Probability

Relationship between $\mathscr{L}_I(s)$ and Link Success

The link success probability for a certain desired SIR threshold γ_{th} can be expressed as

$$\begin{aligned} \mathbb{P}_{suc} &= \Pr\left\{\frac{PHl(r_o)}{\sum_{i\in\Pi_s} H_i l(R_i)P} \geq \gamma_{th}\right\} = \mathbb{E}_I(\exp(-\gamma_{th} r_o^{\alpha} I)) \\ &= \mathscr{L}_I(s)\big|_{s=\gamma_{th} r_o^{\alpha}}. \end{aligned} \tag{18}$$

$$\begin{aligned} \mathscr{L}_I(s) &= \mathbb{E}\left(\exp\left(-s\sum_{i\in\Phi} h_i l(r_i)\right)\right) \\ &= \mathbb{E}_i\left(\prod_{i\in\Phi}\mathbb{E}_H\left[\exp\left(-sh_i l(r_i)\right)\right]\right) \\ &= \exp\left(\int \left(1-\mathbb{E}_H\left[\exp\left(-shl(r)\right)\right]\right)\lambda(dr)\right). \end{aligned} \tag{19}$$

Outline

Stochastic Geometry for spectrum sensing

Exploiting stochastic geometry through two important theorems:

1. Campbell 's theorem
2. Probability generating function

The detection performance at fusion center has been derived analytically (using 1) and the power needed to transmit the measurements to the fusion center has been evaluated theoretically (pgfl)

Cooperative spectrum sensing

1. In practice, several problems militate against effective and efficient spectrum sensing. These include the hidden primary user problem (e.g., inside a large building, etc.), so a local spectrum sensing is not enough.
2. As a result the secondary user cannot detect the primary user and when it accesses this frequency band it will cause interference to the primary user.
3. Because of this, **cooperative spectrum sensing** has emerged to respond to these challenges.

Challenging in cooperative spectrum sensing

- In cooperative spectrum sensing, each secondary user reports its measurement to the fusion center. The reported measurement consumes power and this power consumption might be significant if the number of secondary users is large.
- Thus overhead energy needs to be considered in cooperative spectrum sensing design.
- ***The conventional cooperative*** spectrum sensing is based on censoring the test statistic (local thresold). Here, we use maximum combining at fusion center so we call it censored selection combining **(CSC)**.
- ***To save additional power*** we propose another parameter which is a transmit power threshold along with the local threshold. Thus we call it censored selection combining detector based power censoring **(CSCPC)**.

System Model

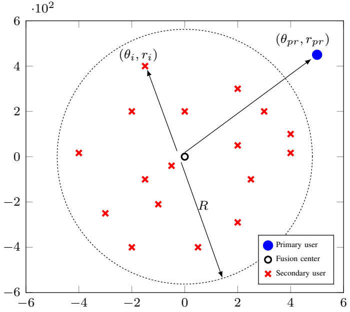


Figure : System model

- Test statistic selection $T_{max} = max_{x \in \Phi} T_x$
- Exploit product form and PGFL.

Average Power Consumption

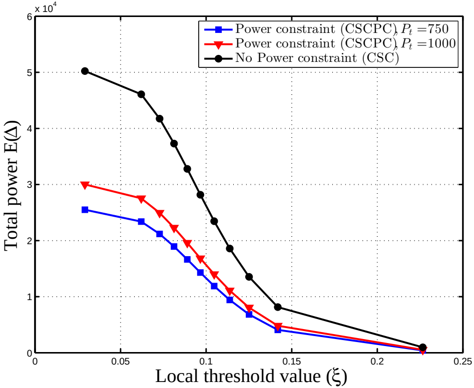


Figure : The average total power $\mathbb{E}[\triangle(\xi, p_t)]$ versus the local threshold (ξ)

Detection Performance

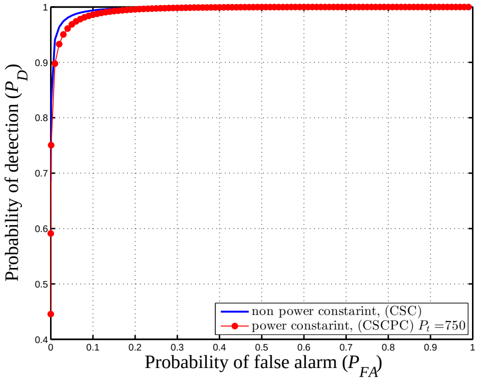


Figure : The probability of detection (P_D) versus the probability of false alarm (P_{FA}) no power constraint (CSC) and for power constraint (CSCPC).

Outline

Two tier network modelling with PPP

- Macro and femtocell positioning as two independent Poisson Point Process Φ_m and Φ_f with intensities λ_m and λ_f respectively.

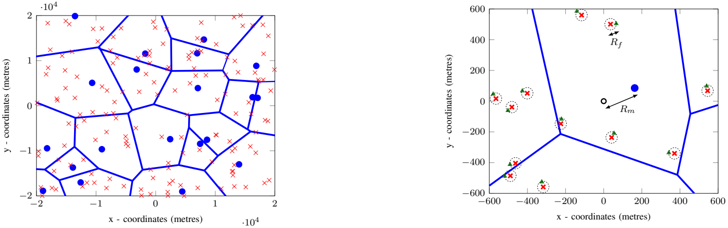


Figure : Two tier network consisting of femtocells (red crosses) and macrocells (blue dots)

- Femtocells are assumed to have a designated user each, considered to be located indoors
- Macrocell users (considered to be located outdoors) are assumed to be served by closest macrocell

System performance metrics

- Signal to Interference Ratio (SIR), for interference limited scenarios

$$SIR_i = \frac{P_i^{tx}|h_0|^2 r_0^{-\alpha_0}|s_0|^2}{\sum\limits_{j\in\Phi_i} P_i^{tx}\left|h_j\right|^2 r_j^{-\alpha_j}\left|s_j\right|^2}$$

where P_i^{tx} is the transmit power of a base station in tier i, $h_0$and h_i are the channel gains (Rayleigh) from serving and interferer base station to desired user, $r_0$and r_j are the distances from user to serving and interfering base station, and s_0 and s_j are the transmitted symbols from serving and interfering base station to the desired user, with $i \in \{f, m\}$

- Coverage probability: Probability that the SIR is above a prescribed threshold β_i

$$P_i^c(\beta_i) = P(SIR_i > \beta_i)$$

- Throughput: Achievable data rates in the system in bps/Hz

$$T_i = P_i^c(\beta_i)\log_2(1+\beta_i)$$

System performance metrics

- Using stochastic geometry concepts, such as Cambpbell's theorem or generating functional, we can find
- Coverage probability

$$P_f^c(\beta_f) = \exp\left(-\lambda_f \left(R_f^{\alpha_0} w \beta_f\right)^{\frac{2}{\alpha_f}} \frac{\pi^2 \frac{2}{\alpha_f}}{\sin\left(\pi \frac{2}{\alpha_f}\right)}\right)$$

$$P_m^c(\beta_m) = \left(1 + \tfrac{2\beta_m}{N_S(\alpha_m - 2)}\, {}_2F_1\left(1, 1 - \tfrac{2}{\alpha_m}; 2 - \tfrac{2}{\alpha_m}; -\beta_m\right)\right)^{-1}$$

where R_f is the femtocell radius, w is the wall partition loss, α_0 is the path loss exponent in the femtocell tier desired link, α_i $(i \in \{f, m\})$ is the path loss exponent of interferers in the i -th tier, ${}_2F_1(a, b; c; d)$ is the Gauss hypergeometric function and N_S is the number of sectors in which macro BS are divided.

System performance metrics

- Throughput: Considering the case of adaptive modulation systems with L constellations available, and integer data rates ($k = 1, 2, .., L$ bps/Hz) the throughput can be expressed as

$$T_f = \sum_{k=1}^{L} P_f^c(\beta_k) = \sum_{k=1}^{L} \exp\left(-\lambda_f \left(R_f^{\alpha_0} w \beta_k\right)^{\frac{2}{\alpha_f}} \frac{\pi^2 \frac{2}{\alpha_f}}{\sin\left(\pi \frac{2}{\alpha_f}\right)}\right)$$

$$T_m = \sum_{k=1}^{L} \left(1 + \frac{2\beta_m}{N_S(\alpha_m - 2)}\ {}_2F_1\left(1, 1 - \frac{2}{\alpha_m}; 2 - \frac{2}{\alpha_m}; -\beta_k\right)\right)$$

Summary

- Spatial point process are essential ingrident of next generation performance evaluation receipe.
- Stochastic geometry provides a formal framework to model the dynamics and capture the interaction in evolving networks.
- Like all other models, stochastic geometric models have certain limitations. However, these can be avoided by intelligently modeling the desired networking scenario.

For Further Reading

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